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THE THERMAL SHOCK AT THE BOUNDARY OF A HALF-SPACE IN THE CASE OF AXIAL SYMMETRY†

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The small-time asymptotic form of the exact solution of the axially symmetric problem of the thermal shock at the boundary of an elastic half-space is obtained. The error in the first two terms of the asymptotic expansion is estimated.

Solutions of the dynamic problem of a thermal shock on the boundary of a half-space were first obtained by Danilovskaya [1, 2]. In subsequent papers, a complete list of which is given in [3, 4], the solution of the one-dimensional problem was extended to different forms of thermal loading. An axially symmetric dynamic thermoelasticity problem in the case of a half-space with a boundary condition of the second kind was considered in [5], where most attention was given to obtaining the asymptotic forms of the displacements as $t \to \infty$.

When investigating the destruction of solids by intense heat, the stressed state of the body at short heating time is of particular interest. It is therefore of particular interest to obtain a simple approximate solution for short times. The use of such a solution is justified if its error is monitored.

An axially symmetry dynamic problem for a half-space is considered below. The methods developed in [6, 7] are used and extended to obtain the approximate solution and to estimate its error.

1. Consider an elastic half-space $z \geqslant 0$ in cylindrical coordinates r, φ and z. Heat transfer from a medium $z < 0$ takes place in accordance with Newton's law on the boundary of this half-space. Up to the instant of time $t = 0$, the half-space and the medium are quiescent at a temperature $T = 0$. At the instant of time $t = 0$, the temperature of the medium increases instantaneously and a distribution

$$\Theta = \Theta_0 f(r) \tag{1.1}$$

is obtained, where the function $f(r)$ allows of a Hankel transform. It is required to find the stresses in the half-space taking account of the dynamic components.

We will change to dimensionless variables by putting

$$T' = \frac{T}{\Theta_0}, \quad r' = \frac{rc_1}{a}, \quad z' = \frac{zc_1}{a}, \quad t' = \frac{tc_1^2}{a}$$

$$\delta' = \frac{\delta c_1}{a}, \quad h' = \frac{ha}{c_1}, \quad \alpha' = \alpha\Theta_0 \tag{1.2}$$

where a is the thermal diffusivity, c_1 is the velocity of longitudinal elastic waves, h is the relative heat transfer coefficient, δ is the characteristic dimension of the distribution $f(r)$ and α is the coefficient of linear expansion. We will henceforth omit the primes when writing dimensionless quantities.

The solution of the heat conduction boundary-value problem

$$\frac{\partial T}{\partial t} = \Delta T, \quad T|_{t=0} = 0, \quad \left.\frac{\partial T}{\partial z}\right|_{z=0} = h(T|_{z=0} - f(r)) \tag{1.3}$$

$$|T(r,z,t)| < \infty \left(\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right)$$

has the form [8]

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$$T^*(r,z,s)=\int_0^\infty \lambda f^H(\lambda)J_0(\lambda r)F(\lambda,z,s)d\lambda,\quad F(\lambda,z,s)=\frac{he^{-\omega z}}{s(\omega+h)}$$

$$T^*(r,z,s)=L_s\{T\}=\int_0^\infty T(r,z,t)e^{-st}dt,\quad f^H(\lambda)=H_\lambda\{f(r)\}=\int_0^\infty rf(r)J_0(\lambda r)dr \tag{1.4}$$

$$\omega=\sqrt{s+\lambda^2},\ \arg\omega=0 \text{ when } s>0$$

(J_n is a Bessel function of the first kind).

In order to determine the thermoelastic displacement potentials it is necessary to solve boundary-value problems for the wave equations

$$\Delta\Phi-\frac{\partial^2\Phi}{\partial t^2}=m_0T,\quad \left(\Delta-\frac{1}{r^2}\right)\Psi-\varepsilon^2\frac{\partial^2\Psi}{\partial t^2}=0$$

$$\Phi|_{t=0}=\left.\frac{\partial\Phi}{\partial t}\right|_{t=0}=\Psi|_{t=0}=\left.\frac{\partial\Psi}{\partial t}\right|_{t=0}=0,\ |\Phi(r,z,t)|<\infty,\ |\Psi(r,z,t)|<\infty \tag{1.5}$$

$$m_0=\frac{1-\nu}{1+\nu}\alpha,\quad \varepsilon^2=c_1^2/c_2^2$$

where ν is Poisson's ratio and c_2 is the velocity of the transverse elastic waves.

The solutions of problems (1.5) are determined using a Laplace transformation with respect to t and Hankel transformation of the zeroth and first order with respect to r. In this case, for the images of the potentials, we obtain

$$\Phi^*(r,z,s)=\int_0^\infty \lambda C(\lambda,s)e^{-R_1z}J_0(\lambda r)d\lambda-m_0\int_0^\infty \lambda f^H(\lambda)J_0(\lambda r)\frac{F(\lambda,z,s)}{R_1^2-\omega^2}d\lambda$$

$$\Psi^*(r,z,s)=\int_0^\infty \lambda D(\lambda,s)e^{-R_2z}J_1(\lambda r)d\lambda \tag{1.6}$$

$$R_1=\sqrt{s^2+\lambda^2},\quad R_2=\sqrt{\varepsilon^2s^2+\lambda^2}$$

$$\arg R_1=\arg R_2=0 \text{ when } s>0$$

Determining the images of the stresses corresponding to Φ^* and Ψ^*, we next find the unknown functions $C(\lambda, s)$, $D(\lambda, s)$ from the boundary conditions

$$\sigma_{zz}^*|_{z=0}=\sigma_{rz}^*|_{z=0}=0 \tag{1.7}$$

Finally, for the images of the required stresses, we obtain

$$\frac{\sigma_{ij}^*}{2m_0G}=\int_0^\infty \lambda f^H(\lambda)[M_{ij}(\lambda,z,s)J_0(\lambda r)+N_{ij}(\lambda,z,s)u_{ij}(\lambda r)]d\lambda,\quad i,j=r,\varphi,z \tag{1.8}$$

$$M_{rr}=M_{\varphi\varphi}=\frac{\nu}{1-2\nu}s^2(\xi_1+\lambda^2\xi_2e_1)-F(\lambda,z,s),\quad M_{zz}=R^2(\xi_1+\lambda^2\xi_2e_{12})$$

$$M_{rz}=0,\quad N_{rr}=N_{\varphi\varphi}=\lambda^2(\xi_1-R^2\xi_2e_2+\lambda^2\xi_2e_1),\quad N_{zz}=0$$

$$N_{rz}=\lambda\left(\frac{R^4\xi_2}{R_2}e_{12}+\omega\xi_1\right),\quad u_{rr}=\frac{J_1(\lambda r)}{\lambda r}-J_0(\lambda r),\quad u_{\varphi\varphi}=-\frac{J_1(\lambda r)}{\lambda r},\quad u_{zz}=0$$

$$u_{rz}=J_1(\lambda r),\quad \xi_1=\frac{h(e^{-R_1z}-e^{-\omega z})}{s(R_1^2-\omega^2)(\omega+h)},\quad \xi_2=\frac{hR_2}{s(R_1+\omega)(\omega+h)P(\lambda,s)}$$

$$R^2 = \varepsilon^2 s^2 / 2 + \lambda^2, \quad P(\lambda, s) = R^4 - \lambda^2 R_1 R_2$$

$$e_1 = e^{-R_1 z}, \quad e_2 = e^{-R_2 z}, \quad e_{12} = e_1 - e_2$$

(G is the shear modulus).

The inverse transforms, corresponding to (1.8), can be formally written down using the inversion theorem.

The practical value of the solution obtained in this manner is obviously small.

2. We will henceforth confine ourselves to functions $f(r)$ for which $f^H(\lambda)$ decay exponentially as λ increases.

Finding the asymptotic expansions of the exact solution as $t \to 0$ reduces to expanding the function $M_{ij}(\lambda, z, s)$, $N_{ij}(\lambda, z, s)$ in series in λ^2, integrating term by term and converting to the inverse transforms. Proof of the asymptotic character of the series obtained in this manner is analogous to that given in [7].

The asymptotic expansion of σ_{zz} has the form

$$\frac{\sigma_{zz}}{2m_0 G} \approx \sum_{k=0}^{\infty} A_{2k+1,0}(r)\varphi_k(z,t), \quad t \to 0$$

$$A_{mn}(r) = \int_0^{\infty} \lambda^m f^H(\lambda) J_n(\lambda r) d\lambda, \quad m = 1,2,..; \; n = 0,1 \tag{2.1}$$

Since $c_1 \sim 10^3$ m/s and $a \sim 10^{-6}$ m²/s, it follows from the fourth relationship of (1.2) that large values of the dimensionless time t' may correspond to physically short times t. Hence, use of the asymptotic form when $t' \to 0$ in order to obtain an approximate solution requires justification. In this connection, we will consider certain features of the asymptotic expansion (2.1). The mappings $\varphi_k^*(z, s) = L_s\{\varphi_k(z, t)\}$ are the coefficients of the expansion, in a power series in λ^2 of the function

$$\varphi^*(\lambda, z, s) = \frac{R^2(e^{-R_1 z} - e^{-\omega z})}{s^2(s-1)(\omega + h)} + \frac{\lambda^2 R^2 R_2 (e^{-R_1 z} - e^{-R_2 z})}{s(R_1 + \omega)(\omega + h)P(\lambda, s)} \tag{2.2}$$

Since the roots of the equation $P(\lambda, s) = 0$ are located on the imaginary axis [9], $\varphi^*(\lambda, z, s)$ has just one singular point $s = 1$ in the half-plane $\operatorname{Re} s > 0$. In the case of the function $\varphi^*(\lambda, z, s)$ and all of its derivatives with respect to λ^2, this singular point is removable.

Actually, for the factor in the first term on the right-hand side of (2.2) which determines the nature of the singular point, we have

$$q(\lambda, z, s) = \frac{e^{-R_1 z} - e^{-\omega z}}{s-1}, \quad \frac{\partial^n q}{\partial(\lambda^2)^n} = \frac{1}{s-1}\left[P_{2n-1}\left(\frac{1}{R_1}\right)e^{-R_1 z} - P_{2n-1}\left(\frac{1}{\omega}\right)e^{-\omega z}\right] \tag{2.3}$$

where $P_m(x)$ is a polynomial of the mth degree in x.

The existence of the finite limits $\lim_{s\to 1}[\partial^n q/\partial(\lambda^2)^n]$, $n = 0, 1, \ldots$ can be demonstrated using (2.3).

According to theorem 35.1 in [10], the behaviour of $\varphi_k(z, t)$ as $t \to \infty$ is determined by the expansion of $\varphi_k^*(z, s)$ in the neighbourhood of the singular point $s = 0$. By investigating the character of the expansion of $\varphi_k^*(z, s)$ in the neighbourhood of $s = 0$, the validity of the relationships

$$\varphi_k(z,t) = t^{2k}\psi_k(z,t) \tag{2.4}$$

can be proved, where $\psi_k(z, t)$ are bounded functions as $t \to \infty$.

We will now return to denoting dimensionless quantities by letters without primes. Since, the function $f(r)$ is dimensionless, $f(r) = f_1(r / \delta) = f_1(r' / \delta')$. By the properties of a Hankel transformation

$$f^H(\lambda) = \delta'^2 f_1^H(\lambda\delta'), \quad A_{mn}(r') = \frac{1}{(\delta')^{m-1}} A_{mn}^{(0)}\left(\frac{r'}{\delta'}\right) \tag{2.5}$$

and $A_{mn}^{(0)}$ have the same values as the dimensional and dimensionless quantities. From (2.1), (2.4) and (2.5), we find

$$\frac{\sigma_{zz}}{2m_0G} \approx \sum_{k=0}^{\infty} \frac{t'^{2k}}{\delta'^{2k}} A^{(0)}_{2k+1,0}\left(\frac{r'}{\delta'}\right)\psi_k(z',t'), \quad t' \to 0 \tag{2.6}$$

Since $t'/\delta' = t_1 = c_1t/\delta$ and the coefficients of t_1^{2k} in the expansion (2.6) are bounded as $t' \to \infty$, it can be expected that the asymptotic form obtained from (2.1), retaining of a finite number of terms, will be sufficiently accurate when $c_1t/\delta \ll 1$. Finally, the quality of the asymptotic form obtained can be established by estimating the error.

Calculations, carried out in connection with the example in Section 4, show that, in the case of the value of t under consideration and certain values of z, the second term of the asymptotic form of σ_{zz} is 8–10 times greater in absolute magnitude than the asymptotic representation of this stress. Hence, unlike the case of a problem with sources [7], the first two terms of each of the asymptotic expansions of σ_{ij} $(i, j = r, \varphi, z)$ are used as the approximate solution. Separating the above-mentioned terms of the asymptotic form, we obtain

$$T(r,z,t) = T^{(0)} + \delta_T = f(r)L_t^{-1}\left\{\frac{he^{-z\sqrt{s}}}{s(\sqrt{s}+h)}\right\} -$$

$$-A_{30}(r)\left[tL_t^{-1}\left\{\frac{he^{-z\sqrt{s}}}{s(\sqrt{s}+h)}\right\} - L_t^{-1}\left\{\frac{he^{-z\sqrt{s}}}{s^2(\sqrt{s}+h)}\right\}\right] + \delta_T$$

$$\frac{\sigma_{jj}}{2m_0G} = -k_jT^{(0)} + l_j\left[f(r)L_t^{-1}\left\{\frac{h(e^{-zs}-e^{-z\sqrt{s}})}{(s-1)(\sqrt{s}+h)}\right\} - A_{30}(r)L_t^{-1}\left\{\frac{hz}{2}\frac{e^{-zs}-\sqrt{s}e^{-z\sqrt{s}}}{s(s-1)(\sqrt{s}+h)} + \right.\right.$$

$$\left.\left. + \frac{h}{2}\frac{e^{-zs}-e^{-z\sqrt{s}}}{\sqrt{s}(s-1)(\sqrt{s}+h)^2} - \frac{4h}{\varepsilon^3}\frac{e^{-zs}}{s^2\sqrt{s}(\sqrt{s}+1)(\sqrt{s}+h)}\right\}\right] -$$

$$-w_j(r)L_t^{-1}\left\{\frac{h(e^{-zs}-e^{-z\sqrt{s}})}{s^2(s-1)(\sqrt{s}+h)} - \frac{2h}{\varepsilon}\frac{e^{-\varepsilon zs}}{s^2\sqrt{s}(\sqrt{s}+1)(\sqrt{s}+h)}\right\} + \delta_{jj}, \quad j = r, \varphi, z \tag{2.7}$$

$$\frac{\sigma_{rz}}{2m_0G} = A_{21}(r)L_t^{-1}\left\{\frac{h(e^{-zs}-e^{-\varepsilon zs})}{s\sqrt{s}(\sqrt{s}+1)(\sqrt{s}+h)} + \frac{h(e^{-zs}-e^{-z\sqrt{s}})}{s\sqrt{s}(s-1)(\sqrt{s}+h)}\right\} -$$

$$-A_{41}(r)L_t^{-1}\left\{\frac{hz}{2}\frac{e^{-zs}-e^{-\varepsilon zs}}{s^2(s-1)(\sqrt{s}+h)} - \frac{hz}{2\varepsilon}\frac{e^{-\varepsilon zs}}{s^2\sqrt{s}(\sqrt{s}+1)(\sqrt{s}+h)} + \right.$$

$$+ \frac{h}{2}\frac{e^{-zs}-e^{-\varepsilon zs}}{s^3(\sqrt{s}+1)(\sqrt{s}+h)} + \frac{h}{2}\frac{e^{-zs}-e^{-\varepsilon zs}}{s^2(\sqrt{s}+1)(\sqrt{s}+h)^2} -$$

$$\left. -\frac{h^2}{2}\frac{e^{-zs}-e^{-z\sqrt{s}}}{s^2\sqrt{s}(s-1)(\sqrt{s}+h)^2} - \frac{4h}{\varepsilon^3}\frac{e^{-zs}-e^{-\varepsilon zs}}{s^3\sqrt{s}(\sqrt{s}+1)(\sqrt{s}+h)}\right\} + \delta_{rz}$$

$$k_r = k_\varphi = 1, \ k_z = 0, \ l_r = l_\varphi = \nu/(1-2\nu), \ l_z = \varepsilon^2/2$$

$w_r = A_{30} - A_{21}/r, \ w_\varphi = A_{21}/r, \ w_z = -A_{30}$, L_t^{-1} is the universe operator of L_s

The inverse transforms in (2.7) are obtained using the tables from [11] and fundamental theorems of the operational calculus.

Relationships (2.7) are exact expressions for the temperature and stresses. The approximate solution is found by discarding the errors δ_T, δ_{ij} $(i, j = r, \varphi, z)$ in (2.7). If, in (2.7), we retain only terms containing $f(r)$ in the expressions for the temperature and normal stresses and terms with the factor $A_{21}(r)$ in σ_{rz}, then we obtain the asymptotic representation of the exact solution. Note that the asymptotic representations of the temperature and normal stresses turn out to be the solutions of the corresponding one-dimensional problems [2] multiplied by $f(r)$.

3. Let us estimate δ_T. From formula (3.3) of [8] and the inequalities $0 < L_t^{-1}\{e^{-z\sqrt{(s)}}(\sqrt{(s)}+h)^{-1}\} \leqslant e^{-z^2/(4t)}(\pi t)^{-1/2}$ [8, 11], we obtain

$$|\delta_T| \leqslant \frac{1}{2}\int_0^\infty \lambda^5 |f^H(\lambda)| d\lambda \int_0^t \tau^2 L_\tau^{-1}\left\{\frac{he^{-z\sqrt{s}}}{\sqrt{s}+h}\right\} d\tau \leqslant \frac{h\beta_5}{5} t^2 \sqrt{\frac{t}{\pi}} e^{-z^2/(4t)} \tag{3.1}$$

$$\beta_n = \int_0^\infty \lambda^n |f^H(\lambda)| d\lambda, \quad n = 1, 2, \ldots$$

For the estimate of δ_{ij} $(i, j = r, \varphi, z)$, we use the formula

$$f_1(\lambda^2, t) * f_2(\lambda^2, t) = f_1(0,t) * f_2(0,t) + \lambda^2[f_1'(0,t) * f_2(0,t) + f_1(0,t) * f_2'(0,t)] +$$

$$+\frac{\lambda^4}{2}[f_1''(\vartheta_1\lambda^2, t) * f_2(\lambda^2, t) + 2f_1'(0,t) * f_2'(0,t) + f_1(\lambda^2, t) f_2''(\vartheta_2\lambda^2, t)] \tag{3.2}$$

$$0 < \vartheta_k < 1, \; k = 1, 2.$$

in which the asterisk is the sign of convolution and differentiation with respect to λ^2 is denoted by primes. Relationship (3.2) is obtained using Taylor's theorem and the properties of a convolution. It is readily generalized to the case of a large number of convolution terms.

As was noted in Section 2, large values of the dimensionless time t' may correspond to physically short times. In order to obtain satisfactory estimates of the errors δ_{ij} $(i, j = r, \varphi, z)$ for sufficiently long intervals of t, one must proceed with some caution in deriving power estimates of the form At^μ. It is desirable to obtain the smallest possible value of μ for which, however, the natural condition $At^\mu = o(\sigma_{ij}^{(k)})$, $t \to 0$ is satisfied ($\sigma_{ij}^{(k)}$ is the last of the terms retained in the asymptotic form of σ_{ij}). These considerations are used to derive the following basic relationships

$$|F_1(\lambda^2, z, t)| = \left|L_t^{-1}\left\{\frac{e^{-R_1 z} - e^{-\omega z}}{s-1}\right\}\right| \leqslant e^{t-z}\eta_0 + \left(\frac{z}{\sqrt{\pi t}} e_z + \lambda^2 z\kappa\right)\eta_1$$

$$|F_1'(0, z, t)| \leqslant \frac{z}{2}\left(\eta_0 + 2z\sqrt{\frac{t}{\pi}} e_z \eta_1 + 2\sqrt{\frac{t}{\pi}} e_z\right)$$

$$F_1''(\lambda^2, z, t) \leqslant t^2\eta_0 + \frac{zt}{2}(t+4)\eta_1, \quad |F_1(\lambda^2, z, t) - F_1(0, z, t)| \leqslant \lambda^2(te^{t-z}\eta_0 + z\kappa\eta_1)$$

$$|F_2(\lambda^2, t)| = \left|L_t^{-1}\left\{\frac{1}{R_1 + \omega}\right\}\right| \leqslant \frac{1}{\sqrt{\pi t}} + 2\lambda, \quad |F_2'(0, t)| \leqslant \frac{t}{2}$$

$$|F_2(\lambda^2, t) - F_2(0, t)| \leqslant \lambda^2 t\sqrt{2}(1 + \sqrt{2t})$$

$$|F_3(\lambda^2, t)| = \left|L_t^{-1}\left\{\frac{sR_1R_2}{P(\lambda, s)}\right\}\right| \leqslant \gamma_1, \quad |F_3(\lambda^2, t) - F_3(0, t)| \leqslant \lambda^2\gamma_2 t$$

$$F_4(\lambda^2, z, t) = L_t^{-1}\left\{\frac{e^{-R_1 z} - e^{-R_2 z}}{R_1}\right\}$$

$$|F_4(\lambda^2, z, t) - F_4(0, z, t)| \leqslant \frac{\lambda^2}{2}[(t^2 - z^2)(\eta_1 - \eta_2) + z(2z + \frac{\varepsilon^2 - 1}{\varepsilon} t)\eta_2 \tag{3.3}$$

$$|F_5(\lambda^2, z, t)| \leqslant \left|L_t^{-1}\left\{\frac{e^{-R_1 z} - e^{-R_2 z}}{s}\right\}\right| \leqslant \eta_1 - \eta_2 + \frac{\lambda^2 z}{2}[(t - z)\eta_1 + (t - \varepsilon z)\varepsilon^{-1}\eta_2]$$

$$|F_5'(0,z,t)| = \frac{z}{2}\left[(t-z)\eta_1 - (t-\varepsilon z)\eta_2\right],\ |F_5''(\lambda^2,z,t)| \le \frac{zt}{12}\left[(t-z)\eta_1 + (t-\varepsilon z)^2\varepsilon^{-3}\eta_2\right]$$

$$|F_5(\lambda^2,z,t) - F_5(0,z,t)| \le \frac{\lambda^2 z}{2}\left[(t-z)\eta_1 + (t-\varepsilon z)\varepsilon^{-1}\eta_2\right]$$

$$F_6(\lambda^2,t) = L_t^{-1}\left\{\frac{1}{\omega+h}\right\} \le F_6(0,t) \le \frac{1}{\sqrt{\pi t}},\quad |F_6'(0,t)| \le \sqrt{\frac{t}{\pi}}$$

$$|F_6''(\lambda^2,t)| \le t\sqrt{\frac{t}{\pi}},\quad |F_6(\lambda^2,t) - F_6(0,t)| \le \lambda^2\sqrt{\frac{t}{\pi}}$$

$$\eta_1 = \eta(t-z),\ \eta_2 = \eta(t-\varepsilon z),\ \eta_0 = 1-\eta_1,\ e_z = \exp(-z^2/(4t)),\ \kappa = 1+\sqrt{t/2}$$

$$\gamma_1 = \frac{2}{\varepsilon^2-1} + \frac{\varepsilon^2-1}{2\pi\varepsilon^2} + \frac{\gamma}{\varepsilon^2},$$

$$\gamma_2 = \frac{\varepsilon^4-1}{8\pi\varepsilon^4} + \frac{\vartheta^2\gamma}{2\varepsilon^4}$$

$$\gamma = \frac{8(\varepsilon^2-\vartheta^2)(1-\vartheta^2)}{\varepsilon^2\vartheta^6 - 6\varepsilon^2\vartheta^4 + (12\varepsilon^2-8)\vartheta^2 - 4(\varepsilon^2-1)}$$

where $\eta(x)$ is the Heaviside unit function and $\pm i\vartheta/\varepsilon$ are the non-zero roots of the equation $P(1,s) = 0$.
We recall that the primes in (3.3) denote differentiation with respect to λ^2.
Methods of deriving relationships (3.3) are given in [6, 7].
We will show how the estimates of δ_{ij} $(i,j = r, \varphi, z)$ are obtained by taking the example of δ_{zz}. From (1.8), we have

$$\begin{aligned}\frac{\sigma_{zz}^*}{2m_0G} &= \frac{\varepsilon^2 h}{2}\int_0^\infty \lambda g(\lambda,r)F_1^*(\lambda^2,z,s)F_6^*(\lambda^2,s)d\lambda + h\int_0^\infty \lambda^3 g(\lambda,r)F_1^*(\lambda^2,z,s)\frac{F_6^*(\lambda^2,s)}{s^2}d\lambda + \\ &+ \frac{\varepsilon^2 h}{2}\int_0^\infty \lambda^3 g(\lambda,r)F_2^*(\lambda^2,s)F_3^*(\lambda^2,s)F_4^*(\lambda^2,z,s)F_6^*(\lambda^2,s)d\lambda + \\ &+ h\int_0^\infty \lambda^5 g(\lambda,r)F_2^*(\lambda^2,s)F_3^*(\lambda^2,s)F_5^*(\lambda^2,z,s)\frac{F_6^*(\lambda^2,s)}{sR_1}d\lambda\end{aligned} \tag{3.4}$$

$$g(\lambda,r) = f^H(\lambda)J_0(\lambda r),\quad F_k^* = L_s\{F_k\},\ k = 1,2,..,6$$

As can be seen from (2.7) and (3.4), the inverse transform of the fourth term in (3.4) occurs in the error δ_{zz}. The asymptotic expansion in (2.7) only contains the first terms of the asymptotic forms as $t \to 0$, of the inverse transforms of the second and third terms in (3.4). The errors, corresponding to the last three terms in (3.4), are estimated using the methods from [7] on the basis of relationships (3.3).
Using (3.2), we obtain

$$\begin{aligned}\delta_{zz}^{(1)} &= \frac{\varepsilon^2 h}{4}\int_0^\infty \lambda^5 g(\lambda,r)\left[F_1''(\vartheta_1\lambda^2,z,t) * F_6(\lambda^2,t) + 2F_1'(0,z,t) * F_6'(0,t) + \right. \\ &\left. + F_1(\lambda^2,z,t) * F_6''(\vartheta_6\lambda^2,t)\right]d\lambda,\quad 0 < \vartheta_m < 1,\ m = 1,6\end{aligned} \tag{3.5}$$

for the inverse transform of the first term in (3.4) and the terms of the asymptotic expansion in (2.7) corresponding to it. The final estimate of $\delta_{zz}^{(1)}$ is obtained from (3.5) by evaluating the convolutions and integrals with respect to λ.

On carrying out similar operations for the remaining errors, we finally obtain

$$|\delta_{jj}| \leqslant h\beta_5\left\{\frac{k_j}{5\sqrt{\pi}}t^{5/2}e_z + l_j\left[\frac{1}{5\sqrt{\pi}}p_1\left(\frac{5}{2}\right) + z\left(\frac{3}{16}e_z + \frac{\kappa\beta_7}{5\beta_5}\sqrt{\frac{t-z}{\pi}}\right)(t-z)^2\eta_1 + \right.\right.$$

$$\left. + \frac{1}{\sqrt{\pi}}p_1\left(\frac{1}{2}\right)p_3(2) + \frac{zt}{2}(t+4)\sqrt{\frac{t-z}{\pi}}\eta_1 + \frac{z}{\sqrt{\pi}}p_1\left(\frac{3}{2}\right) + z\left(\frac{t^2}{8} + \frac{2z\sqrt{t}}{3\pi}(t-z)^{3/2}\eta_1\right)e_z\right] +$$

$$+\left(1-\frac{k_j}{2}\right)\left[\frac{8}{15\sqrt{\pi}}\left(p_1\left(\frac{5}{2}\right)p_3(1) + z\kappa(t-z)^{5/2}\eta_1\right) + \frac{2}{7\sqrt{\pi}}p_1\left(\frac{7}{2}\right) + \frac{z}{\pi}e_z(t-z)^3\eta_1 + \right.$$

$$\left. + \frac{\gamma_1}{24}p_2(4) + \frac{64\gamma_1}{945\sqrt{\pi}}\frac{\beta_6}{\beta_5}p_2\left(\frac{9}{2}\right) + \frac{z\gamma_1}{6}\left(\frac{\beta_7}{8\beta_5}p_4(5) + \frac{64}{315\sqrt{\pi}}\frac{\beta_8}{\beta_5}p_4\left(\frac{11}{2}\right)\right)\right] +$$

$$+(1-k_j)l_j\gamma_1\left[\frac{1}{12}p_2(3) + \frac{16\sqrt{2}}{105\sqrt{\pi}}p_2\left(\frac{7}{2}\right) + \left(\frac{\sqrt{\pi}}{16} + \frac{\gamma_2}{12\gamma_1}\right)p_2(4) + t(t-z)\left(\frac{1}{2}p_2(2) + \right.\right.$$

$$\left.\left. + \frac{8}{15\sqrt{\pi}}\frac{\beta_6}{\beta_5}p_2\left(\frac{5}{2}\right)\right) + \frac{zt}{2\varepsilon}(\varepsilon^2+1)\left(\frac{1}{2} + \frac{8\beta_6}{15\beta_5}\sqrt{\frac{t-\varepsilon z}{\pi}}\right)(t-\varepsilon z)^2\eta_2\right] +$$

$$\left. +k_j\frac{\varepsilon}{2(1-2\nu)}\left[\left(\frac{\gamma_1}{9} + \frac{\gamma_2}{12}\right)p_4(4) + \gamma_2\left(\frac{1}{8} + \frac{4t}{15\sqrt{\pi}}\frac{\beta_6}{\beta_5}\right)p_4\left(\frac{7}{2}\right) + \frac{\gamma_1}{4}\left(t+\frac{1}{3}\right)p_4(3)\right]\right\}$$

$j = r, \varphi, z$

$$|\delta_{rz}| \leqslant \frac{h\beta_7 r}{4}\left\{\frac{zt}{36}\left[p_4(3) + \frac{64}{35\sqrt{\pi}}\frac{\beta_8}{\beta_7}p_4\left(\frac{7}{2}\right) + \left(\frac{1}{8} + \frac{z}{6}\right)p_2(3) + \right.\right.$$

$$\left. + \frac{32}{105\sqrt{\pi}}\left(5 + \frac{3\beta_8}{\beta_7} + 2z\right)p_2\left(\frac{7}{2}\right) + \frac{7}{25\sqrt{\pi}}(t^2+8)^{3/4}\left[\frac{2}{5}p_2\left(\frac{13}{4}\right) + \frac{z}{12}\frac{\beta_9}{\beta_7}p_4\left(\frac{17}{4}\right)\right]\right] +$$

$$+\frac{z}{8}\left(\frac{\beta_9}{8\beta_7} + \frac{\gamma_1}{3}\right)p_4(4) + \frac{1}{2}p_1(2)p_3(2) + \frac{8}{315\sqrt{\pi}}\left[4z\left(\frac{\beta_{10}}{\beta_7} + \frac{\gamma_1\beta_8}{3\beta_7}\right)p_4\left(\frac{9}{2}\right) + \right. \tag{3.6}$$

$$\left. +3hzp_1\left(\frac{7}{2}\right) + hp_1\left(\frac{9}{2}\right) + 3hz(3+2z)\sqrt{\frac{t}{\pi}}e_z(t-z)^{7/2}\eta_1 + hz\kappa\frac{\beta_9}{\beta_7}(t-z)^{9/2}\eta_1\right] +$$

$$\left. + \frac{zt}{4}(t+4)(t-z)^2\eta_1 + \frac{hz}{48}t^4 + \frac{\gamma_1}{24}p_2(4) + \frac{64\gamma_1}{945}\sqrt{\frac{2}{\pi}}p_2\left(\frac{9}{2}\right) + \frac{1}{10}\left(\frac{\gamma_1\sqrt{\pi}}{4} + \frac{\gamma_2}{3}\right)p_2(5)\right\}$$

$$p_1(x) = t^x - (t-z)^x\eta_1, \quad p_2(x) = (t-z)^x\eta_1 - (t-\varepsilon z)^x\eta_2$$

$$p_3(x) = t^x\eta_0 + z^x\eta_1, \quad p_4(x) = (t-z)^x\eta_1 + (t-\varepsilon z)^x\varepsilon^{-1}\eta_2$$

4. We will now consider an example. Let $f(r) = \delta^3(r^2+\delta^2)^{-3/2}$. Then $f^H(\lambda) = \delta^2 e^{-\lambda\delta}$, A_{mn} can be expressed in terms of elementary functions [8], $\beta_n = n!/\delta^{n-1}$.

Calculations using (2.7) were carried out for $\nu = 0.25$, $\delta = 4\times10^9$. When $t = 1.5\times10^8$, $r = 0.2\delta$, the dimensionless stress $\sigma_{22}/(2m_0G)$ has a minimum $\sigma_{\min}^{(1)} = -2.64\times10^{-6}$ when $z = t$ (on the longitudinal wavefront), a maximum $\sigma_{\max} = 1.409\times10^{-9}$ when $\zeta = 0.6306t/\varepsilon$ ($z = t/\varepsilon$ is the transverse wavefront) and a minimum $\sigma_{\min}^{(2)} =$

-5.622×10^{-10} when $z = 0.0004t$. The derivative $\partial\sigma_{zz}/\partial z$ has a discontinuity on the longitudinal wavefront. The maximum absolute error δ_{zz}, by (3.6), does not exceed 8.625×10^{-11}. The relative error when $0 \leqslant z \leqslant t$ does not exceed 8% outside the neighbourhoods of values of z in which σ_{zz} changes sign.

The stress $\sigma_{rz}/(2m_0G)$, which is characteristic in the case of a non-one-dimensional problem, has a minimum $\sigma_{\min}^{(1)} = -7.471 \times 10^{-13}$ when $z = 0.8828t$, a maximum $\sigma_{\max} = 5.245 \times 10^{-12}$ when $z = t/\varepsilon$ and a minimum $\sigma_{\min}^{(2)} = -2.890 \times 10^{-8}$ when $z = 0.0006t$. The maximum relative error when $0 \leqslant z \leqslant t$ does not exceed 15% outside the neighbourhood of values of z in which σ_{rz} changes sign.

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